Journal of Statistical Physics, Vol. 59, Nos. 1/2, 1990

Cluster Expansion for *d*-Dimensional Lattice Systems and Finite-Volume Factorization Properties

Enzo Olivieri[1,2] and Pierre Picco[1]

Received October 24, 1988; revision received October 31, 1989

We consider classical lattice systems with finite-range interactions in d dimensions. By means of a block-decimation procedure, we transform our original system into a polymer system whose activity is small provided a suitable factorization property of finite-volume partition functions holds. In this way we extend a result of Olivieri.

KEY WORDS: *d*-dimensional lattice gas; cluster expansion; analyticity; finite-volume mixing.

1. INTRODUCTION

This paper is concerned with the statistical mechanics of classical lattice spin systems in d dimensions. By means of a block-decimation procedure, we introduce a cluster expansion which is convergent provided a suitable finite-size condition is fulfilled. In this way we extend to the general d-dimensional situation the results already found in ref. 1 for the two-dimensional case. We refer to this last paper for a more complete discussion of the motivations and the interest of our approach.

Here we only remark on the differences with respect to the standard high-temperature expansions (see, for instance, ref. 2).

(i) The basic length scale in terms of which the geometrical objects of the expansion are defined is one ($\equiv$the spacing of the lattice) in the standard theories, whereas it is a free parameter L in ours.

(ii) The reference system around which one performs the perturba-

---

[1] Centre de Physique Théorique, Marseille, France.

[2] Permanent address: Dipartimento di Matematica, Università dell'Aquila, L'Aquila, Italy.

0022-4715/90/0400-0221$06.00/0

tion (cluster) expansion is universal (e.g., is a system of independent spins) in the usual theories; in our approach it consists of a set of finite-volume systems independent of each other but with nontrivial correlations in their interior.

(iii) The small parameter in the usual high-temperature of low-activity expansion is just the inverse temperature or the activity or some simple combination of them; here it is related to the mixing properties of a finite-volume system. We do not need, *a priori*, that all the interactions between the microscopic constituents of our system be small, but, rather, we take advantage of the thermal averages in order to exploit the weak effective interaction between regions that are sufficiently far apart with respect to the correlation length.

For the above reasons we believe that our approach can be useful, for instance, for treating ferromagnetic systems in the "intermediate-temperature" region, namely in the region of temperatures $[T_c, T_0]$, where T_c is the critical temperature, $T_c \equiv \inf\{T\text{: spontaneous magnetization } m^*(T)=0\}$, and T_0 is the estimated threshold of absolute convergence of the usual high-temperature expansion.

Our results imply the absence of any kind of phase transition; so if our condition for a *finite-volume* system can be verified (for example, by means of a computer), then from its validity one can deduce rigorous consequences about the corresponding *infinite-volume* system.

We think that our condition can be useful for a computer-assisted proof of the absence of a phase transition even though to get significant improvements of the region of convergence with respect to the traditional cluster expansions one probably needs very powerful computers.

Our results are strictly related to the analogous ones by Dobrushin and Shlosman. In ref. 3–6 these authors developed a theory of the uniqueness and analyticity of the Gibbs states which does not make use of the cluster expansion. In ref. 6 the authors give 12 equivalent conditions that ensure that a given system belong to the class of the so-called completely analytical interactions.

These conditions are "constructive" in the sense that they only need to be verified in some finite volume V whose size depends on the constants involved in the conditions themselves (see ref. 6 for more details). As remarked in ref.1, the Dobrushin–Shlosman conditions imply ours and it is very easy to see that the present paper constitutes, in particular, an alternative proof of the results of ref. 6 (see Remark 2.1).

In other words, Theorem 1.1 below says that we have found another equivalent constructive condition for an interaction to belong to the class of completely analytical potentials. We think this result of some interest not only because it possibly gives a better and easier-to-verify condition, but also because its proof, which is based on a very different approach, seems to be shorter and, in our opinion, more transparent.

Moreover, we think that our approach can be applied to some problems generally treated via high-temperature expansions, such as the Ornstein–Zernike behavior of the two-point correlation function. Finally, we want to stress that our approach allows us to obtain a finite-size condition for the convergence involving thermal averages (see Theorem 1.2) instead of the supremum over the boundary conditions (see Theorem 1.1). Of course, one cannot avoid using this kind of condition in mean in order to treat systems of unbounded spins. On the other hand, our weaker condition can be useful also for Monte Carlo simulations. One can "measure" with a Monte Carlo method the quantities involved in our sufficient condition and state (if it is satisfied) that one is in the pure phase region. In other words, one can perform an "empirical" test for the validity of a condition which *rigorously* implies the absence of phase transitions for the infinite-volume system.

Let us now define the model and state the results.

Given $\Lambda \subset \mathbb{Z}^d$, the configuration space in Λ is the set $S_\Lambda = \{0, 1,..., s\}^\Lambda$ for some fixed integer s. A configuration in Λ is a map $\sigma: \Lambda \to \{0, 1,..., s\}$. We denote by $|\Lambda|$ the cardinality of a finite set $\Lambda \subset \mathbb{Z}^d$, and for $x \equiv (x_1;...; x_d)$, $y \equiv (y_1;...; y_d)$ in $\mathbb{Z}^d$ we denote by $\text{dist}(x, y)$ or $\|x-y\|$ the distance defined as

$$\|x-y\| = \max_{i=1,...,d} |x_i - y_i|$$

If $\Lambda \subset \mathbb{Z}^d$, $\text{diam}\,\Lambda \equiv \sup_{x,y \in \Lambda \times \Lambda} \|x-y\|$ is the diameter of Λ. We suppose given a potential $U = \{U_X;\, X \subset \mathbb{Z}^d,\, |X| < \infty\}$, $U_X: S_X \to \mathbb{R}$ such that:

(C1) $\exists r_0 > 0$: $U_X = 0$ if $\text{diam}\, X > r_0$ (finite range).

(C2) $\forall X \subset \mathbb{Z}^d$, $|X| < \infty$, $\forall y \in \mathbb{Z}^d$, $U_{X+y} = U_X$ (translation invariance).

Given a finite volume $\Lambda \subset \mathbb{Z}^d$, we denote by

$$H_\Lambda(\sigma_\Lambda) = \frac{-1}{T} \sum_{x \subset \Lambda} U_X(\sigma_x) \tag{1.1}$$

the energy associated with the generic spin configuration (σ_Λ) in S_Λ multiplied by $-1/T$ (T being the temperature).

Given two disjoint finite regions Λ_1, Λ_2 in $\mathbb{Z}^d$, we define the interaction between Λ_1 and Λ_2 by

$$W_{\Lambda_1,\Lambda_2}(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}) = H_{\Lambda_1 \cup \Lambda_2}(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}) - H_{\Lambda_1}(\sigma_{\Lambda_1}) - H_{\Lambda_2}(\sigma_{\Lambda_2}) \tag{1.2}$$

Given a finite volume $\Lambda \subset \mathbb{Z}^d$, we call the "outer boundary" of Λ the set

$$\partial_{r_0}\Lambda = \{x \in \mathbb{Z}^d \backslash \Lambda: \text{dist}(x, \Lambda) \leqslant r_0\}$$

Given a spin configuration $\beta_{\partial_{r_0}\Lambda} \in S_{\partial_{r_0}\Lambda}$, the finite-volume Gibbs measure with boundary condition β is given by

$$\mu_\Lambda^\beta(\sigma_\Lambda) = Z(\Lambda; \beta)^{-1} \exp[H_\Lambda(\sigma_\Lambda) + W_{\Lambda, \partial_{r_0}\Lambda}(\sigma_\Lambda, \beta_{\partial_{r_0}\Lambda})] \tag{1.3}$$

where

$$Z(\Lambda; \beta) = \sum_{\sigma_\Lambda \in S_\Lambda} \exp H_\Lambda(\sigma_\Lambda) + W_{\Lambda, \partial_{r_0}\Lambda}(\sigma_\Lambda, \beta_{\partial_{r_0}\Lambda}) \tag{1.4}$$

is the partition function in Λ with boundary conditions β. We often say that Λ and β are respectively the support and the boundary conditions of the partition function $Z(\Lambda; \beta)$ and we write

$$\Lambda = \text{supp } Z(\Lambda, \beta) \tag{1.5}$$

We set, $\forall \beta$,

$$Z(\emptyset; \beta) \equiv 1 \tag{1.6}$$

Now let $Q_L(x)$ be the cube of edge L centered at x, for a given $x \in \mathbb{Z}^d$ and an odd integer L; namely

$$Q_L(x) = \left\{ y \in \mathbb{Z}^d \colon \|x - y\| \leqslant \frac{L-1}{2} \right\}$$

Let $Q_{3L} \equiv Q_{3L}(0)$ be the cube of edge $3L$ and center the origin of $\mathbb{Z}^d$.

We set

$$Q_{3L} = Q_{3L}^-(j) \cup Q_{3L}^0(j) \cup Q_{3L}^+(j)$$

where

$$Q_{3L}^{0,+,-}(j) = \{ y \in Q_L(x),\ Q_L(x) \subset Q_{3L},\ (x)_j = 0,\ +L,\ -L \} \tag{1.7}$$

where $j \in \{1, 2, ..., d\}$ and $(x)_j$ is the jth component of x.

In other words, we divide Q_{3L} into three slices according to a given direction of the lattice.

Let L be an odd integer $> r_0$. Let $P_{L,j}$ be the set of all subsets of $Q_{3L}^0(j)$ which (i) are unions of cubes $Q_L(x)$, (ii) contain $Q_L(0)$, and (iii) are symmetric with respect to all the directions of the lattice different from the jth one.

Let σ_+, σ_- be spin configurations $\in S_{Q_{3L}^+(j)}$, $S_{Q_{3L}^-(j)}$ and τ a spin configuration $\in S_{Q_{3L}^0(j) \setminus \Lambda}$; we denote by $Z^{(j)}(\Lambda; \sigma_-, \sigma_+, \tau)$ the partition function with support $\Lambda \in P_{L,j}$ and boundary conditions $\sigma_{-,+}$ in $Q_{3L}^{-,+}(j)$, respectively, τ in $Q^0_{3L}\backslash\Lambda$ and 0, namely the configuration identically equal to 0 in Q^c_{3L}. Notice that here 0 plays only the role of a fixed reference configuration.

Condition C_L. For a given L the following inequality holds:

$$\sup_{\sigma_-,\sigma_+,\tau} \sup_{j} \sup_{\Lambda\subset P_{L,j}} \left| \frac{Z^{(j)}(\Lambda;\sigma_-,\sigma_+,\tau)\, Z^{(j)}(\Lambda;0,0,\tau)}{Z^{(j)}(\Lambda;\sigma_-,0,\tau)\, Z^{(j)}(\Lambda;0,\sigma_+,\tau)} - 1 \right|$$
$$< [3(2^{d+1}+1)]^{-d}\cdot 2^{-2d}e^{-4} \tag{1.8}$$

The following theorem contains our main result.

Theorem 1.1. Suppose that conditions C1 and C2 are satisfied and that there exists an L such that the corresponding condition C_L holds. Given $U^1,..., U^l$ translationally invariant real potentials with finite range $\leqslant r_0$, consider the complex partition function $Z(\Lambda;\beta;\tilde{U})$ defined as in Eq. (1.4) with $\tilde{U}=U+\sum_{j=1}^{l}\lambda_j U^j$ in place of U and $\lambda_j\in\mathbb{C}$, $j=1,..., l$. Let $f^\beta_\Lambda:\mathbb{C}^l\to\mathbb{C}$ be given by

$$f^\beta_\Lambda(\lambda_1,..., \lambda_l)=\frac{1}{|\Lambda|}\log Z(\Lambda,\beta,\tilde{U}) \tag{1.9}$$

Then there exists a neighborhood Ω of 0 in $\mathbb{C}^l$ such that, for $\lambda\equiv(\lambda_1,..., \lambda_l)\in\Omega$, the limit

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\log Z(\Lambda;\beta;\tilde{U})$$

exists and is a holomorphic function of $\lambda_1,..., \lambda_l$ in Ω.

In fact, the thesis of the previous theorem could have been, immediately after Eq. (1.9), "Then, there is a convergent cluster expansion for the free energy."

The above analyticity result is quite standard to deduce when expression (2.5.6) and Proposition 2.5.2 below hold true.

Moreover, in this situation one can prove uniqueness of the Gibbs state as well as more general analyticity results also concerning thermal averages of local observables even with respect to nontranslationally invariant complex perturbations. Finally, exponential decay of truncated correlation functions can also be proven. We omit the complete proof of Theorem 1.1 and we just prove Proposition 2.5.2, of which, using Proposition 2.5.3, it is an easy corollary.

Looking at the proof of Proposition 2.5.1, it is not hard to see that our results can be extended to the case of long-range interactions that decay with the distance r at least as $1/r^{2d+\varepsilon}$ (where d is the dimension and $\varepsilon > 0$). In fact, one can adapt to the general d-dimensional case the methods developed in refs. 7 and 8. The case of general compact spin systems does not present any particular new difficulty and a result similar to Theorem 1.1 can be obtained along the same lines.

The announced stronger result is contained in the following Theorem 1.2. For simplicity, we state it in the same hypotheses as those of Theorem 1.1, but it is clear that one can obtain a similar result also for the case of unbounded spin systems. The statement of Theorem 1.2 needs many preliminary definitions and it is only to give a simpler and self-contained statement that we have given the weaker form of the result contained in Theorem 1.1.

Theorem 1.2. Suppose that conditions C1 and C2 are satisfied and that there exists an L such that the corresponding condition C'_L of Section 2 below is satisfied. Then the statement of Theorem 1.1 holds true.

Again the complete proof is omitted. It is an easy consequence of Propositions 2.5.2 and 2.5.4.

2. POLYMERIZATION AND CLUSTER EXPANSION

The main purpose of the present section is to transform our original spin system into a polymer system by performing a block-decimation procedure. Since this section is rather long, we now give a short description of it. In Section 2.1 we introduce a partition of $\mathbb{Z}^d$ into 2^d disjoint sublattices and make the d-dimensional analogs of a regular pavement with 2^d different d-dimensional cubes. In Section 2.2 we describe the first of the 2^d steps of our block-decimation procedure. Section 2.3 is the heart of the paper; we define there the basic operations that we will perform in order to continue our procedure of summation. In Section 2.4 we describe the result of the general step of summation. In Section 2.5 we state the main result of this paper, namely we define the polymer system we have obtained and perform a cluster expansion.

2.1. Notation and Geometric Considerations

Let L be an odd integer, $L > r_0$ being the range of interaction. This L will be our fundamental length scale; all objects we shall define live on this scale. We denote by $\hat{\mathbb{Z}}^d$ the original lattice where our spin system is defined. In order to introduce our partition of $\hat{\mathbb{Z}}^d$ into cubes of side L, we use an auxiliary lattice $\mathbb{Z}^d$ whose points are in one-to-one correspondence with the centers of the cubes of side L in $\hat{\mathbb{Z}}^d$. For $x \in \mathbb{Z}^d$ we denote by $Q(x)$ the cube with center in Lx and side L: $Q(x) \equiv Q_L(Lx) = \{y \in \mathbb{Z}^d \colon \|y - Lx\| \leqslant L - \frac{1}{2}\}$.

Moreover, for any subset V of $\mathbb{Z}^d$ we set $Q(V) = \bigcup_{x \in V} Q(x)$; in particular, $\hat{\mathbb{Z}}^d = Q(\mathbb{Z}^d)$. We often identify a set of points V in $\mathbb{Z}^d$ with the union of the unit cubes centered at the points of V. In this way $Q(V)$ becomes the homothetic image of V in $\hat{\mathbb{Z}}^d$. Now, given a partition of $\mathbb{Z}^d$ into 2^d disjoint sublattices of spacing 2, $\mathbb{Z}^d = \bigcup_{k=1}^{2^d} \mathbb{Z}_k^d$, our partition of $\hat{\mathbb{Z}}^d$ into cubes of side L will be given by

$$\hat{\mathbb{Z}}^d = \bigcup_{k=1}^{2^d} Q(\mathbb{Z}_k^d) \tag{2.1.1}$$

Our partition of $\mathbb{Z}^d$ is defined in the following way.

$\mathbb{Z}_1^d$ is simply $2\mathbb{Z}^d$, i.e., the sublattice of $\mathbb{Z}^d$ of spacing 2 centered at the origin. For $1 < k \leqslant 2^d$ we take $\mathbb{Z}_k^d$ as the sublattice of spacing 2 obtained from $\mathbb{Z}_{k-1}^d$ with a translation by a unit vector e_k parallel to one of the lattice directions; namely, the points of $\mathbb{Z}_k^d$ are nearest neighbors of $\mathbb{Z}_{k-1}^d$ in $\mathbb{Z}^d$ (see Fig. 1 in the case $d = 2$). In the sequel we often identify a block $Q(x)$ in $\hat{\mathbb{Z}}^d$ by its center x in $\mathbb{Z}^d$ and we denote by $p(x)$ the index of the sublattice to which its belongs; more precisely, $p \colon \mathbb{Z}^d \to \{1, 2, ..., 2^d\}$ is given, for $x \in \mathbb{Z}^d$, by $p(x) = j$ if $x \in \mathbb{Z}_j^d$.

Given $x \in \mathbb{Z}^d$, we write $\partial x = \{y \in \mathbb{Z}^d \mid \|x - y\| = 1\}$; $D(x) \equiv x \cup \partial x$ will denote the cube in $\mathbb{Z}^d$ with center at x and edge 3. More generally, given $V \in \mathbb{Z}^d$, we write $\partial V = \{x \in \mathbb{Z}^d \mid \operatorname{dist}(x, V) = 1\}$.

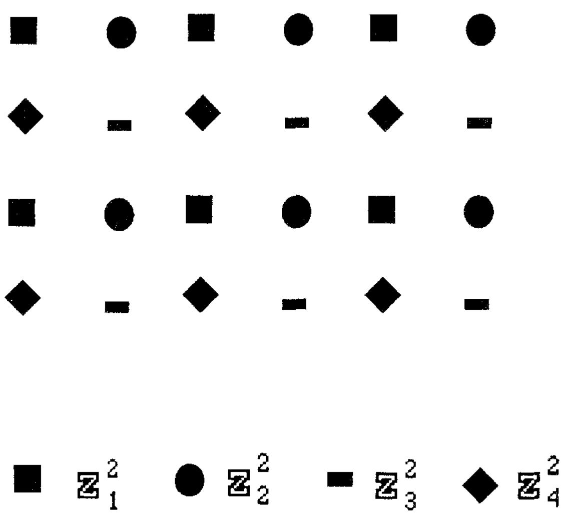


Fig. 1.

Now, if $k \in 1,..., 2^d$, we denote by Γ_k the family of parallel hyperplanes of dimension $d-1$, orthogonal to e_{k+1}, passing through points x_k of $\mathbb{Z}_k^d$; more precisely,

$$\Gamma_k = \bigcup_{n \in \mathbb{Z}} Y(x_k + 2ne_{k+1}; e_{k+1}) \tag{2.1.2}$$

where x_k is an arbitrary point of $\mathbb{Z}_k^d$ and we have denoted by $Y(x; e)$ the hyperplane of dimension $d-1$ passing through x and orthogonal to e. We denote by $\Gamma_k + e_{k+1}$ the translate of the set Γ_k by the vector e_{k+1}. In this way we get the foliation of $\mathbb{Z}^d$ in hyperplanes orthogonal to the direction e_{k+1} (see Fig. 2), that is,

$$\mathbb{Z}^d = \Gamma_k \cup (\Gamma_k + e_{k+1}) \tag{2.1.3}$$

We always think of the spin configurations in a finite volume $\Lambda \subset \hat{\mathbb{Z}}^d$ as spin configurations in the whole $\hat{\mathbb{Z}}^d$ by simply extending them to 0 in $\mathbb{Z}^d \backslash \Lambda$ and we use the notation $\sigma(\Lambda)$ instead of the more usual σ_Λ to denote a spin configuration with support in Λ:

$$\sigma(\Lambda): \quad \mathbb{Z}^d \to \{0,..., n\}$$

such that

$$\sigma(\Lambda)(x) = 0, \qquad \forall x \in \mathbb{Z}^d \backslash \Lambda$$

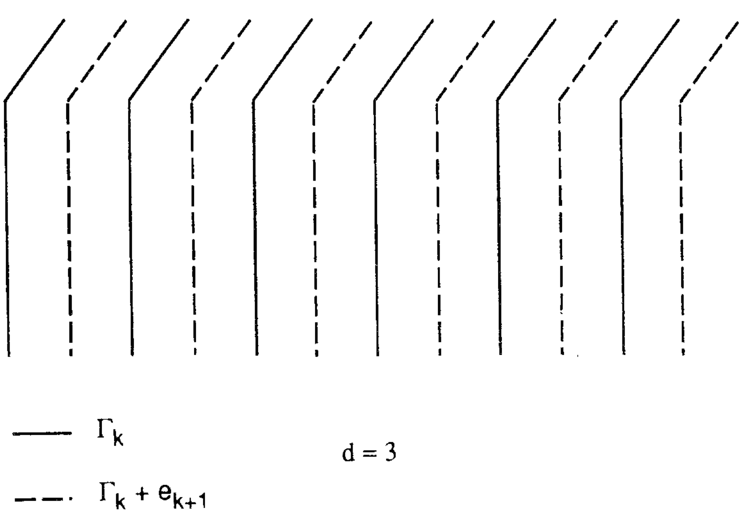


Fig. 2.

Given A and A' subsets of $\hat{\mathbb{Z}}^d$, and $\sigma(A)$ and $\tau(A')$ two spin configurations, we say that they are compatible if $\sigma(A \cap A') = \tau(A \cap A')$ with the convention that if $A \cap A' = \emptyset$, they are always compatible.

Given two compatible configurations $\sigma(A)$ and $\tau(A')$, we denote by $\sigma(A) \wedge \tau(A')$ the spin configuration which coincides with σ (and τ) on $A \cap A'$ and with 0 on $(A \cap A')^c$ and we denote by $\sigma(A) \vee \tau(A')$ the spin configuration which coincides with σ on A, τ on A', and 0 on $(A \cup A')^c$.

If $x \in \mathbb{Z}^d_k$, a spin configuration in $Q(x)$ will be denoted by $\alpha_k(x)$ and a spin configuration in $Q(\mathbb{Z}^d_k)$ will be simply denoted by α_k.

2.2. The First Step

Let m be an integer and $V_m \equiv V \subset \mathbb{Z}^d$ the cube of side $2m+1$ centered at the origin; we consider a spin system enclosed in the cube $\Lambda = Q(V) \subset \mathbb{Z}^d$. For simplicity we consider periodic boundary conditions; it will be clear from what follows that with minor changes, any boundary condition can be treated.

Let $H(\alpha_k(x)) \equiv H_{Q(x)}(\alpha_k(x))$ be the self-energy of a block centered at Lx for some $x \in \mathbb{Z}^d_k$. If $k \in \{1,..., 2^d\}$, we shall write $V_k = V \cap \mathbb{Z}^d_k$;

$$H_V(\alpha_k) \equiv \sum_{x \in V_k} H(\alpha_k(x)) \tag{2.2.1}$$

is the self-energy of the family of blocks centered on the kth sublattice and belonging to $Q(V)$.

If $x \in V_k$, the interaction energy of a block $Q(x)$ with those neighboring blocks which have an index strictly bigger than k is given by

$$W_V(\alpha_k(x); \alpha_{>k}) = W_{Q(x);\, \bigcup_{h=k+1}^{2^d} \bigcup_{x_h \in \partial x \cap V_h} Q(x_h)} \left(\alpha_k(x);\ \bigvee_{h=k+1}^{2^d} \ \bigvee_{x_h \in \partial x \cap V_h} \alpha_h(x_h) \right) \tag{2.2.2}$$

The corresponding interactions for the volume V_k will be denoted by

$$W_V(\alpha_k; \alpha_{>k}) = \sum_{x \in V_k} W_V(\alpha_k(x); \alpha_{>k}) \tag{2.2.3}$$

Using the partition given by (2.1.1), we can write

$$H_\Lambda(\sigma(\Lambda)) = \sum_{k=1}^{2^d} H_V(\alpha_k) + W_V(\alpha_k; \alpha_{>k}) \tag{2.2.4}$$

Now we want to perform a block-decimation procedure by summing first on the α_1 variables, then on the α_2, and so on.

If we denote by $\sum_{\alpha_k}$ the sum over all configurations in $S_{Q(A_k)}$, we can write

$$Z_A = \sum_{\alpha_1,\dots,\alpha_{2^d}} e^{H_A(\alpha(A))}$$
$$= \sum_{\alpha_{2^d}} e^{H_V(\alpha_{2^d})} \sum_{\alpha_{2^d-1}} e^{H_V(\alpha_{2^d-1}) + W_V(\alpha_{2^d-1};\alpha_{>2^d-1})}$$
$$\cdots \sum_{\alpha_k} e^{H_V(\alpha_k) + W_V(\alpha_k;\alpha_{>k})}$$
$$\cdots \sum_{\alpha_1} e^{H_V(\alpha_1) + W_V(\alpha_1;\alpha_{>1})} \tag{2.2.5}$$

Using the fact that the range of the interaction is smaller than L, we get for a fixed configuration $\alpha_2,\dots,\alpha_{2^d}$:

$$\sum_{\alpha_1} e^{H_V(\alpha_1) + W_V(\alpha_1;\alpha_{>1})} = \prod_{x_1 \in V_1} Z(A_1(x_1); \beta_1(x_1)) \tag{2.2.6}$$

where

$$Z(A_1(x_1); \beta_1(x_1)) = Z(Q(A_1(x_1)), \beta(Q(\partial A_1(x_1))) \tag{2.2.7}$$

where $A_1(x_1) = \{x_1\}$ and $\beta_1(x_1)$ is the spin configuration in $Q(\partial x_1)$; namely $\beta_1(x_1) = \{\alpha_k(x_k); k \geq 2, x_k \in \partial x_1 \cap \mathbb{Z}^d_k\}$.

As is clear from expression (2.2.6), after the first summation over α_1 the term $Z(A_1(x_1); \beta_1(x_1))$ couples the configurations in $Q(x_1 + e_2)$ and $Q(x_1 - e_2)$, giving rise, in such a way, to an effective interaction between them.

In a sense that we are going to make precise, if our finite-size factorization property C_L holds, this effective interaction becomes small.

Definition 2.2.1. Given $\varepsilon \in \{-1, 0, +1\}$, $x \in \mathbb{Z}^d$, and a unit vector e parallel to one of the lattice directions, we define a map

$$S^{\varepsilon}_{x,e}\colon \quad S_{Q(D(x))} \to S_{Q(D(x))} \tag{2.2.8}$$

in the following way:

(i) If $\varepsilon \subset \{-1, 0, +1\}$ and if $z \in Y(x; e) \cap D(x)$,

$$S^{\varepsilon}_{x,e}(\sigma(Q(z))) = \sigma(Q(z)) \tag{2.2.9}$$

(ii) For $\varepsilon = 0$ and $z \notin Y(x, e) \cap D(x)$,

$$S^0_{x,e}(\sigma(Q(z))) = 0 \tag{2.2.10}$$

where 0 means the 0 configuration in $Q(z)$.

(iii) If $\varepsilon \in \{-1, +1\}$ and $z \in Y(x+\varepsilon_1 e; e\} \cap D(x)$ for some $\varepsilon_1 \in \{-1, +1\}$,

$$S^{\varepsilon}_{x,e}(\sigma(Q(z))) = \begin{cases} \sigma(Q(z)) & \text{if } \varepsilon_1 = \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{2.2.11}$$

Here also 0 means the zero configuration in $Q(z)$.

We define also, if Λ is an arbitrary subset of $D(x)$,

$$S^{0}_{x,e}\Lambda = \Lambda \cap Y(x; e) \tag{2.2.12}$$

and if $\varepsilon \in \{+1 -1\}$,

$$S^{\varepsilon}_{x,e}\Lambda = \Lambda \cap \{Y(x, e) \cup Y(x+\varepsilon e, e)\} \tag{2.2.13}$$

Now if $Z(\Lambda; \beta)$ is a partition function with support $\Lambda \subset D(x)$ and boundary condition β, we define

$$S^{\varepsilon}_{x,e}Z(\Lambda; \beta) = Z(S^{\varepsilon}_{x,e}\Lambda; S^{\varepsilon}_{x,e}\beta) \tag{2.2.14}$$

where, also here, we set $Z(\varnothing; \beta) = 1$.

Let us now write for any $x_1 \in V_1$ the following trivial identity, where for simplicity $Z(\Lambda_1(x_1), \beta_1(x_1))$ will be denoted by Z:

$$Z = \frac{(S^{+}_{x_1,e_2}Z)(S^{-}_{x_1,e_2}Z)}{S^{0}_{x_1,e_2}Z}(1+\Phi^1_{x_1}) \tag{2.2.15}$$

where

$$\Phi^1_{x_1} = \frac{(Z)(S^{0}_{x_1,e_2}Z)}{(S^{+}_{x_1,e_2}Z)(S^{-}_{x_1,e_2}Z)} - 1 \tag{2.2.16}$$

is the function

$$\Phi^1_{x_1}: \quad S_{Q(\partial x_1)} \to \mathbb{R}$$

$$\Phi^1_{x_1} = \Phi^1_{x_1}(\{\alpha_k(x_k); k \geqslant 2, x_k \in \partial x_1 \cap \mathbb{Z}^d_k\})$$

We notice that since the dependence on $\{\alpha_2(x_1+e_2), \alpha_2(x_1-e_2)\}$ in $S^{+}_{x_1,e_2}Z$ and $S^{-}_{x_1,e_2}Z$ is factorized, the only interaction between $Q^{(2)}_L(x_1+e_2)$ and $Q^{(2)}_L(x_1-e_2)$ is present in the term $\Phi^1_{x_1}$. This is small by virtue of the finite-size factorization property (C_L). If we disregard the "error terms" $\Phi^1_{x_1}$, the partition function $Z(\Lambda_1(x_1), \beta_1(x_1))$ becomes factorized and we can easily perform the summation over the α_2 variables so that we can repeat the same procedure. In this way we would produce new error terms and again we could try to continue the procedure by disregarding them.

2.3. The General Step

In each one of our 2^d steps we will operate over all sublattices at the same time. As will be clear in the following, a step will not merely consist in the summation over the actual variables, it will rather consist in the following operations that we are going to define: (i) unfolding, (ii) splitting, (iii) gluing, and (iv) summation over the spin variables.

Definition 2.3.1. Given Λ_1 and Λ_2 two subsets of $\mathbb{Z}^d$, $\sigma_1(\Lambda_1^c)$ and $\sigma_2(\Lambda_2^c)$ two compatible configurations, we call *unfolding* the substitution of the left-hand side of the following identity by its right-hand side:

$$Z(\Lambda_1 \cup \Lambda_2; \sigma_1 \vee \sigma_2) = \frac{Z(\Lambda_1; \sigma_1)\, Z(\Lambda_2; \sigma_2)}{Z(\Lambda_1 \cap \Lambda_2; \sigma_1 \wedge \sigma_2)} (1 + \Phi) \tag{2.3.1}$$

where

$$\Phi = \frac{Z(\Lambda_1 \cup \Lambda_2; \sigma_1 \vee \sigma_2)\, Z(\Lambda_1 \cap \Lambda_2; \sigma_1 \wedge \sigma_2)}{Z(\Lambda_1; \sigma_1)\, Z(\Lambda_2; \sigma_2)} - 1 \tag{2.3.2}$$

In particular, we call unfolding of a partition function $Z(\Lambda; \sigma)$ at the point x in the direction e the previous substitution in the case where

$$\Lambda_1 = S_{x,e}^{+} \Lambda, \qquad \Lambda_2 = S_{x,e}^{-} \Lambda, \qquad \sigma_1 = S_{x,e}^{+} \sigma, \qquad \sigma_2 = S_{x,e}^{-} \sigma$$

Now we define the reciprocal operation.

Definition 2.3.2. Given Λ_1 and Λ_2 two subsets of $\mathbb{Z}^d$, $\sigma_1(\Lambda_1^c)$ and $\sigma_2(\Lambda_2^c)$ two compatible configurations, we call *gluing* the following substitution:

$$\frac{Z(\Lambda_1; \sigma_1)\, Z(\Lambda_2; \sigma_2)}{Z(\Lambda_1 \cap \Lambda_2; \sigma_1 \wedge \sigma_2)} = Z(\Lambda_1 \cup \Lambda_2; \sigma_1 \vee \sigma_2)(1 + \Phi) \tag{2.3.3}$$

where

$$\Phi = \frac{Z(\Lambda_1; \sigma_1)\, Z(\Lambda_2; \sigma_2)}{Z(\Lambda_1 \cap \Lambda_2; \sigma_1 \wedge \sigma_2)\, Z(\Lambda_1 \cup \Lambda_2; \sigma_1 \vee \sigma_2)} - 1 \tag{2.3.4}$$

In particular, we call gluing at the point $x \in \mathbb{Z}^d$ in the direction e the previous substitution in the case where

$$\Lambda_1 \subset D(x-e) \cap D(x), \qquad \Lambda_2 \subset D(x) \cap D(x+e)$$

and

$$\sigma((D(x-e) \cap D(x)) \backslash \Lambda_1), \qquad \sigma'((D(x) \cap D(x+e)) \backslash \Lambda_2)$$

are two compatible configurations.

Let us remark that if $\Lambda_1 \cap \Lambda_2 = \emptyset$, the gluing is degenerate in the sense that $Z(\Lambda_1 \cup \Lambda_2; \sigma \vee \sigma') = Z(\Lambda_1; \sigma)\, Z(\Lambda_2; \sigma')$, $\Phi = 0$, and $Z(\Lambda_1 \cap \Lambda_2; \sigma \wedge \sigma') = 1$.

Definition 2.3.3. Let $j \in \{1, ..., 2^d\}$ and, for any $x \in V_j$, let $\Lambda(x)$ be a subset of $D(x)$ and $\beta_+(x)$ [resp. $\beta_-(x)$] be a spin configuration in $(D(x) \cap D(x+e)) \backslash \Lambda(x)$ [resp. $(D(x) \cap D(x-e)) \backslash \Lambda(x)$] for some vector $e \in \{e_k\}_{k=2}^{2^d}$; we call *splitting* in the direction e the following substitution which is induced by the changes of variable $x \to y = x \pm e$:

$$\prod_{x \in V_j} Z(\Lambda(x) \cap D(x+e), \beta_+(x))\, Z(\Lambda(x) \cap D(x-e), \beta_-(x))$$
$$= \prod_{y \in V_i} Z(\Lambda(y-e) \cap D(y), \beta_+(y-e))\, Z(\Lambda(y+e) \cap D(y), \beta_-(y+e)) \tag{2.3.5}$$

where $i = p(x+e)$ with $x \in V_j$.

Let us remark that we only perform a splitting in the following cases:

$$\Lambda(x) \cap D(x+e) = S^+_{x,e} \Lambda(x)$$
$$\Lambda(x) \cap D(x-e) = S^-_{x,e} \Lambda(x)$$

$$\beta_+ = S^+_{x,e}\beta, \qquad \beta_- = S^-_{x,e}\beta \quad \text{for some spin configuration in } D(x)\backslash \Lambda(x)$$

To show how the previous definitions are useful and to introduce the general step, we now perform a summation on the α_2 variables.

Let us call, starting from Eq. (2.2.6),

$$\tilde{Z}_2(\alpha \geqslant 3) = \sum_{\alpha_2} \prod_{X_2 \in V_2} e^{H(\alpha_2(x_2)) + W(\alpha_2(x_2);\, \alpha_{>2})} \times \prod_{X_1 \in V_1} Z(\Lambda_1(x_1); \beta_1(x_1)) \tag{2.3.6}$$

We perform an unfolding at all points $x_1 \in V_1$ in the direction e_2 and we get

$$\tilde{Z}_2(\alpha_{\geqslant 3}) = \sum_{\alpha_2} \prod_{x_2 \in V_2} e^{H(\alpha_2(x_2)) + W(\alpha_2(x_2);\, \alpha_{>2})}$$
$$\times \prod_{x_1 \in V_1} \frac{[S^+_{x_1, e_2} Z(\Lambda_1(x_1); \beta_1(x_1))][S^-_{x_1, e_2} Z(\Lambda_1(x_1); \beta_1(x_1))]}{S^0_{x_1, e_2} Z(\Lambda_1(x_1); \beta_1(x_1))}$$
$$\times \prod_{x_1 \in V_1} (1 + \Phi^1_{x_1}) \tag{2.3.7}$$

Now we perform a splitting in the direction e_2, that is, we write

$$\begin{aligned}\prod_{x_1 \in V_1} &[S^+_{x_1,e_2} Z(A_1(x_1); \beta_1(x_1))][S^-_{x_1,e_2} Z(A_1(x_1); \beta_1(x_1))]\\ &= \prod_{x_2 \in V_2} [S^+_{x_2-e_2,e_2} Z(A_1(x_2-e_2); \beta_1(x_2-e_2))]\\ &\quad \times [S^-_{x_2+e_2,e_2} Z(A_1(x_2+e_2); \beta_1(x_2+e_2))] \qquad (2.3.8)\end{aligned}$$

where we have used $\mathbb{Z}^d_2 = \mathbb{Z}^d_1 + e_2$.

If all the $\Phi^1_{x_1}$ were equal to zero, by construction, we would obtain

$$\begin{aligned}\tilde{Z}_2(\alpha_{\geqslant 3})|_{\phi^1 \equiv 0} &= \sum_{\alpha_2} \prod_{x_2 \in V_2} e^{H(\alpha_2(x_2)) + W(\alpha_2(x_2); \alpha_{>2})}\\ &\quad \times \prod_{x_2 \in V_2} [S^+_{x_2-e_2,e_2} Z(A_1(x_2-e_2); \beta_1(x_2-e_2))]\\ &\quad \times [S^-_{x_2+e_2,e_2} Z(A_1(x_2+e_2); \beta_1(x_2+e_2))]\\ &\quad \times \prod_{x_1 \in V_1} [S^0_{x_1,e_2} Z(A_1(x_1); \beta_1(x_1))]^{-1}\\ &= \prod_{x_1 \in V_1} [S^0_{x_1,e_2} Z(A_1(x_1); \beta_1(x_1))]^{-1}\\ &\quad \times \prod_{x_2 \in V_2} \Bigg\{ \sum_{\alpha_2(x_2)} e^{H(\alpha_2(x_2)) + W(\alpha_2(x_2); \alpha_{>2})}\\ &\quad \times [S^+_{x_2-e_2,e_2} Z(A_1(x_2-e_2); \beta_1(x_2-e_2))]\\ &\quad \times [S^-_{x_2+e_2,e_2} Z(A_1(x_2+e_2); \beta_1(x_2+e_2))] \Bigg\} \qquad (2.3.9)\end{aligned}$$

where we have used the fact that $S^0_{x_1,e_2} Z(A_2(x_1), \beta_1(x_1))$ does not depend on α_2.

It can be easily checked that each factor in the last product in the rhs of Eq. (2.3.9) is nothing but the partition function with support $Q(\{x_2\} \cup \{x_2+e_2\} \cup \{x_2-e_2\}) \equiv Q(A_2(x_2))$ and boundary conditions

$$\beta_2(x_2) = \begin{cases} \sigma(Q(y)) & \text{if } \; y \in \partial x_2,\ p(y) \geqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, by setting

$$Z(A_2(x_2); \beta_2(x_2)) = Z(Q(A_2(x_2)); \beta_2(x_2))$$

we get, by neglecting the Φ's,

$$\tilde{Z}_2(\alpha_{\geqslant 3})|_{\phi^1\equiv 0} = \prod_{x_2\in V_2} Z(A_2(x_2);\beta_2(x_2)) \times \prod_{x_1\in V_1} [S^0_{x_1,e_2} Z(A_1(x_1);\beta_1(x_1))]^{-1} \tag{2.3.10}$$

In the real case where $\phi^1_{x_1}$ is nonzero, by multiplying and dividing by the factor

$$\prod_{x_2\in V_2} Z(A_2(x_2);\beta_2(x_2))$$

we get

$$\tilde{Z}_2(\alpha_{\geqslant 3}) = \prod_{x_2\in V_2} Z(A_2(x_2);\beta_2(x_2)) \times \prod_{x_1\in V_1} [S^0_{x_1,e_2} Z(A_1(x_1);\beta_1(x_1))]^{-1} \times \nu^2_{\alpha>2}\left(\prod_{x_1\in V_1}(1+\Phi^1_{x_1})\right) \tag{2.3.11}$$

where $\nu^2_{\alpha>2}$ is the normalized product measure on $S_{Q(V_2)}$ defined by

$$\nu^2_{\alpha>2}(f) = \sum_{\alpha_2} f(\alpha_2)\, \nu^2_{\alpha>2}(\alpha_2)$$

if $f\colon S_{Q(V_2)} \to \mathbb{R}$ with

$$\nu^2_{\alpha>2}(\alpha_2) = \prod_{x_2\in V_2} e^{H(\alpha_2(x_2)) + W(\alpha_2(x_2);\,\alpha>2)} \times \frac{S^+_{x_2-e_2,e_2} Z(A_1(x_2-e_2);\beta_1(x_2-e_2)) \times S^-_{x_2+e_2,e_2} Z(A_1(x_2+e_2);\beta_1(x_2+e_2))}{Z(A_2(x_2);\beta_2(x_2))} \tag{2.3.12}$$

Let us now describe the general step. We give here a rough description of the sequence of operations that we perform without giving exact definitions and without any proof. The precise statements will be contained in the rest of the present section and in Section 3.

Our procedure can be described as a sort of "cellular automaton" in the sense that at each time we apply simultaneously at each site (of $\mathbb{Z}^d$, the lattice that characterizes the centers of the cubes of our partition) a transformation different from site to site. The variables present at each site, at each any given time, will be a volume, a boundary condition, and a variable taking value in $\{-1, +1\}$. The first two correspond respectively to the support and to the boundary condition of a partition function and the third one will say if this partition function is raised to be power -1 or $+1$. The criterion for updating our variables in a given site at a given time will be local, in the sense that it will only depend on the variables on the neighboring sites, and nonstationary in the sense that it will depend explicitly on time. This "time evolution" corresponds to writing the same quantity (the global partition function) in different ways by summing at each time over a new class of variables corresponding to one of the 2^d sublattices of the partition (2.1.1).

After the summations on the spin configuration $\alpha_k,..., \alpha_1$ for some $k \geqslant 2$, we claim to have produced beyond other terms a product of the form

$$\prod_{x \in V} [Z_k(x)]^{\varepsilon_k(x)}$$

where $\varepsilon_k(x) \in \{+1, -1\}$ and $Z_k(x)$ are partition functions whose supports are indexed by x. For simplicity we ignore for the moment the important question of boundary conditions.

Moreover, some of the previous partition functions could be equal to one if their support is the empty set (if $k=2$, this corresponds to all $x \notin V_1 \cup V_2$). This is a spurious effect which occurs at the beginning of our procedure of summation, but there exists an integer $k_0 \in \{1,..., 2^d\}$ such that if $k > k_0$, all the partition functions are different from 1.

Now, for any $x \in \Gamma_k \cap V$, we perform an unfolding at the point x in the direction e_{k+1}. If $Z_k(x) = 1$, this is a completely trivial operation. Let us remark that we perform an unfolding even in the case when $\varepsilon_x(x) = -1$.

Then we perform a splitting in the direction e_{k+1}. In particular, at this stage we get, ignoring the error terms, for any $x \in \Gamma_k + e_{k+1}$ a factor like

$$[\tilde{Z}_k^0(x)]^{\tilde{\varepsilon}_k^{(1)}(x)} [\tilde{Z}_k^+(x) \tilde{Z}_k^-(x)]^{+\tilde{\varepsilon}_k^{(2)}(x)}$$

where the $\tilde{Z}_k^+(x) \tilde{Z}_k^-(x)$ comes from the previous splitting, whereas the $\tilde{Z}_k^0(x)$ are already present, coming from the previous steps. In some cases $\tilde{Z}_k^0 = 1$.

The main difficulty of our construction is to realize that at this stage we are in the right situation to perform a gluing at all points $x \in \Gamma_k + e_{k+1}$, namely, we have to have obtained at each point an expression like

$$\left(\frac{Z(\Lambda_1) Z(\Lambda_2)}{Z(\Lambda_1 \cap \Lambda_2)} \right)^{\pm 1}$$

with analogous compatibility conditions for the boundary conditions. In the following subsection we will introduce tools that allow us to prove this nontrivial fact. Now, assuming that it is possible, we perform a gluing at all points $x \in \Gamma^k + e_{k+1} \cap V$, then we perform the summation over the spin configurations α_{k+1}. An important remark at this stage is that if one ignores the error terms, the dependence on α_{k+1} variables is factorized so that one can perform independently the sum over $\alpha_{k+1}(x_{k+1})$ for any $x_{k+1} \in \mathbb{Z}^d_{k+1}$, and get in this way, for any $x_{k+1} \in \mathbb{Z}^d_{k+1}$, a new partition function. We stress that the effect of the unfolding, splitting, and gluing is also present in the other sublattices.

What we really do, taking into account that $\Phi \neq 0$, is similar to what we did to obtain (2.3.11), namely, we multiply and divide by the proper normalization factor and we get, in this way, a normalized product measure over α_{k+1} variables.

2.4. The Result of a Generic Step

In this subsection we define all supports and boundary conditions of the partition functions together with the measures and error terms that we have obtained at the kth step of our procedure.

The following definition corresponds, when $k=2$, to the first term on the right-hand side of Eq. (2.3.11).

Definition 2.4.1. Given $k \in 1, 2, 3, \ldots, 2^d$, we define for any $x_k \in \mathbb{Z}^d_k$

$$\begin{aligned} A_k(x_k) &= \{x \in D(x_k);\ p(x) \leqslant k\} \\ B_k(x_k) &= \{x \in \partial(x_k);\ p(x) > k\} \end{aligned} \tag{2.4.1}$$

Given a generic spin configuration σ, we define

$$\beta_k(x_k) = \begin{cases} \sigma(Q(B_k(x_k))) & \text{in} \quad Q(B_k(x_k)) \\ 0 & \text{otherwise} \end{cases} \tag{2.4.2}$$

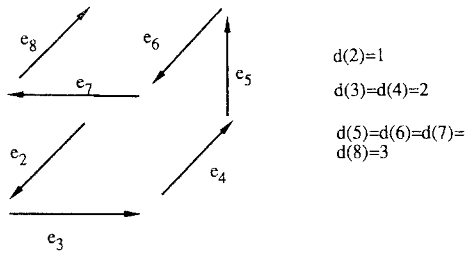
$$\varepsilon_k(x_k) = +1$$

See Fig. 3 for the case $k=3$, $d=3$ and Fig. 4 for the case $k=5$, $d=3$.

Now we want to define the support, the boundary conditions of all partition functions we have produced, together with the exponent at which they arise.

Given $k \in 2, \ldots, 2^d$, let $\mathbb{Z}^{d(k)}$ be the sublattice of $\mathbb{Z}^d$ of dimension $d(k)$ with spacing 1 centered at the origin, generated by the vectors $e_2, \ldots, e_k$. We set

$$M^{d(k)} = \bigcup_{j=1}^{2^{d(k)}} \mathbb{Z}^d_j$$


e8
e6
e7
e5
e2
e4
e3
d(2)=1
d(3)=d(4)=2
d(5)=d(6)=d(7)=
d(8)=3

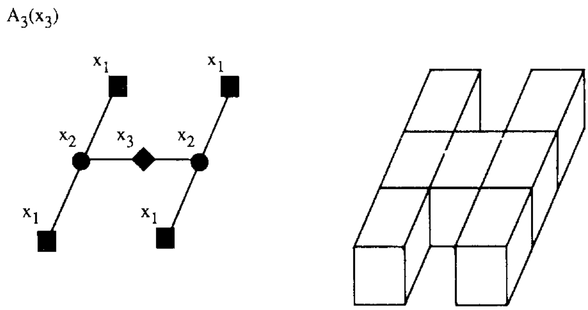

A3(x3)
x1
x1
x2
x3
x2
x1
x1

Fig. 3.

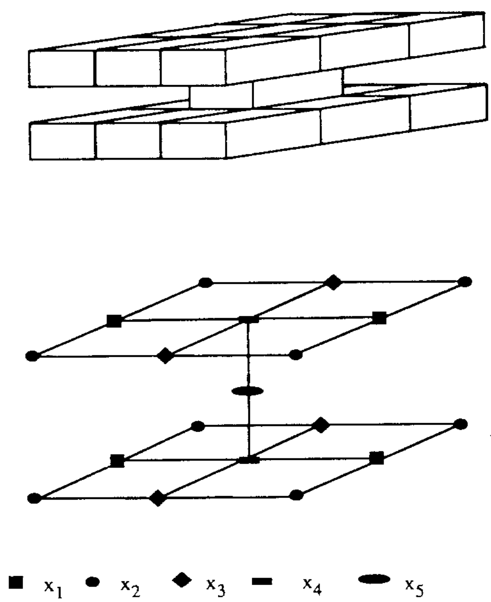

x1
x2
x3
x4
x5

Fig. 4.

Notice that $M^d = \mathbb{Z}^d$. In our previous description of the general step the points x which belong to $\mathbb{Z}^d \backslash M^{d(k)}$ are precisely those for which $Z = 1$.

Let $u_1, ..., u_{d(k)}$ be a *base* of vectors, parallel to the axis of $\mathbb{Z}^d$, generating $\mathbb{Z}^{d(k)}$.

Now, given $j \in \{1, 2, ..., 2^{d(k)}\}$, let $d(k, j)$ be the distance between the two sublattices $\mathbb{Z}^d_k$ and $\mathbb{Z}^d_j$ induced by the following metric on $\mathbb{Z}^d$: $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i|$. Moreover, we can find among $u_1, ..., u_{d(k)}$ a family of orthogonal vectors $\{v_l\}_{l=1,...,d(k,j)}$ together with a family of signs $\{\varepsilon_l\}_{l=1,...,d(k,j)}$ such that

$$\mathbb{Z}^d_j = \mathbb{Z}^d_k + \sum_{l=1}^{d(k,j)} \varepsilon_l \cdot v_l$$

This will be denoted by $\mathbb{Z}^d_j = \mathbb{Z}^d_k + \gamma(k, j)$ and we write $|\gamma(k, j)| = d(k, j)$. We say that a vector e is orthogonal to $\gamma(k, j)$ if $e \perp v_l \, \forall l \in \{1, ..., |\gamma(k, j)|\}$; this will be denoted by $e \perp \gamma(k, j)$.

Definition 2.4.2. Given $k \in 1, ..., 2^d$ and $j \in \{1, 2, ..., 2^{d(k)}\}$, we define

$$Y(x; \gamma(k, j)) = \bigcap_{l=1}^{|\gamma(k,j)|} Y(x; v_l) \tag{2.4.3}$$

Namely, $Y(x; \gamma(k, j))$ is the affine hyperplane of codimension $|\gamma(k, j)|$ orthogonal to the vectors $v_1, ..., v_{|\gamma(k,j)|}$ passing through x.

Now, if $Z(A_k(x), \beta_k(x))^{\varepsilon_k(x)}$ is the partition function that appears at x after k summations, we have the following.

Definition 2.4.3. Given $k \in 1, ..., 2^d$ and $x \in \partial x_k$, $x_k \in \mathbb{Z}^d_k$ with $p(x) = j$ for some $j \in 1, ..., 2^d$, we define

$$A_k(x) = \begin{cases} \varnothing & \text{if} \quad p(x) > 2^{d(k)} \\ A_k(x_k) \cap Y(x; \gamma(k, j)) & \text{otherwise} \end{cases} \tag{2.4.4}$$

$$B_k(x) = B_k(x_k) \cap Y(x; \gamma(k, j))$$

Given a generic spin configuration σ, we write

$$\beta_k(x) = \begin{cases} \sigma(Q(B_k(x))) & \text{in} \quad Q(B_k(x)) \\ 0 & \text{otherwise} \end{cases} \tag{2.4.5}$$

$$\varepsilon_k(x) = (-1)^{|\gamma(k,j)|}$$

See Fig. 5 for the case $k = 3$, $j = 1, 2$, $d = 3$; and Fig. 6 for the case $k = 5$, $j = 4, 3, 2, 1$, $d = 3$.

Now we define the normalized measures obtained in our process of summation.

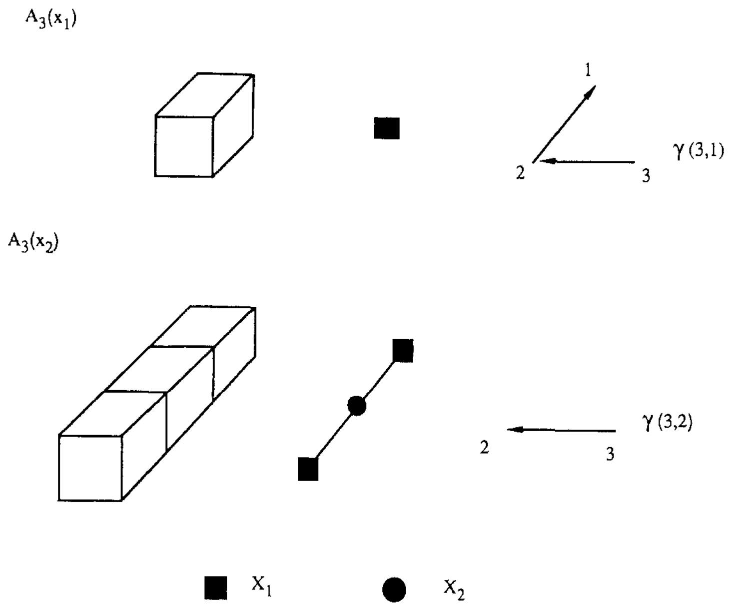


Fig. 5.

Definition 2.4.4. Given $k \in 2, 3,..., 2^d$ and a spin configuration $\alpha_{k+1},..., \alpha_{2^d}$ on $\bigcup_{j=k+1}^{2^d} V_j$, we define the normalized Bernoulli measure $\nu^k_{\alpha>k}$ on S_{V_k} by

$$\nu^k_{\alpha>k}(f) = \sum_{\alpha_k} f(\alpha_k)\, \nu^k_{\alpha>k}(\alpha_k) \tag{2.4.6}$$

where $f: S_{V_k} \to \mathbb{R}$,

$$\nu^k_{\alpha>k}(\alpha_k) = \prod_{x_k \in V_k} \nu^k_{\beta_k(x_k)}(\alpha_k(x_k))$$

and for any $x_k \in V_k$,

$$\nu^k_{\beta_k(x_k)}(\alpha_k(x_k)) = \frac{e^{H(\alpha_k(x_k)) + W(\alpha_k(x_k);\, \beta_k(x_k))}}{Z(A_k(x_k);\, \beta_k(x_k))}\, Z(A_k(x_k) \backslash x_k;\, \beta_k(x_k))$$

We denote by

$$\nu^{k+1}_{\alpha>k+1} \circ \nu^k_{\alpha>k}$$

the measure on $S_{V_{k+1} \cup V_k}$ with weights

$$\nu^{k+1}_{\alpha>k+1}(\alpha_{k+1})\, \nu^k_{\alpha>k}(\alpha_k)$$

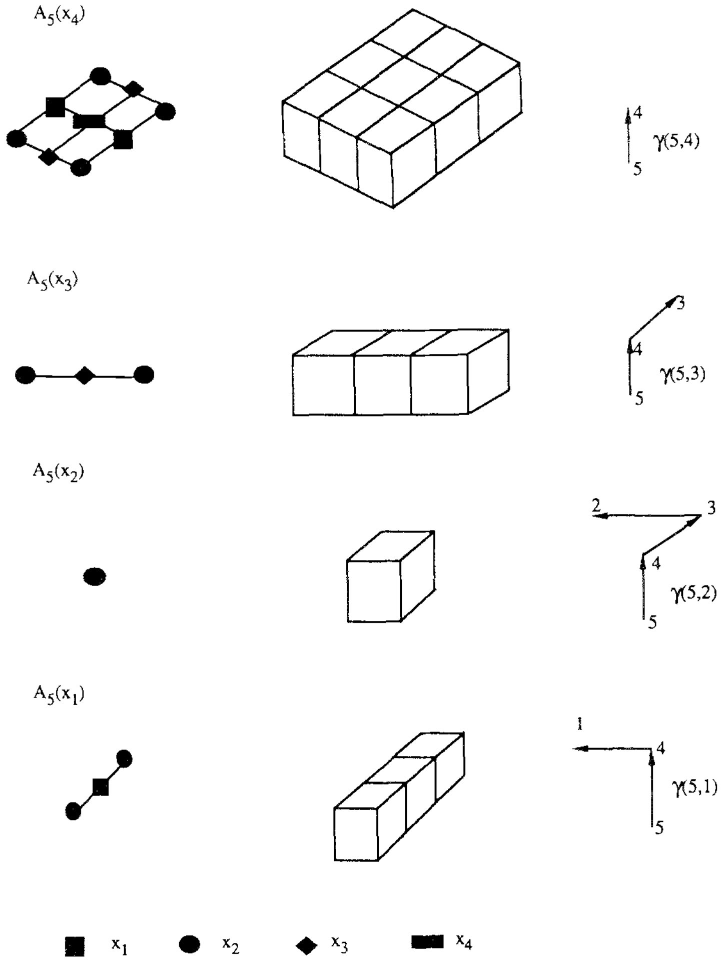


Fig. 6.

We emphasize that the weights $\nu^k_{\alpha>k}(\alpha_k)$ depend on α_{k+1} and therefore $\nu^{k+1}_{\alpha>k+1} \circ \nu^k_{\alpha>k}$ is not a product measure, but rather a composition of the two measures.

Now we define the error terms we produce when we perform an unfolding [Eq. (2.4.7)] or a gluing [Eq. (2.4.8)].

Definition 2.4.5. For any $k \in 1,..., 2^d$ and $x \in \Gamma_k$ we define

$$\Phi^k_x = -1 + \left[\frac{Z(A_k(x); \beta_k(x))\, Z(A_{k+1}(x); \beta_{k+1}(x))}{S^+_{x,e_{k+1}} Z(A_k(x); \beta_k(x))\, S^-_{x,e_{k+1}} Z(A_k(x); \beta_k(x))}\right]^{\varepsilon_k(x)} \qquad (2.4.7)$$

If $x \in \Gamma_k + e_{k+1}$ and $x \notin V_{k+1}$,

$$\begin{aligned}\Phi_x^k = & -1 + [Z(A_k(x); \beta_k(x))\, Z(A_{k+1}(x); \beta_{k+1}(x))] \\ & \times [S^+_{x-e_{k+1}, e_{k+1}} Z(A_k(x - e_{k+1}); \beta_k(x - e_{k+1})) \\ & \times S^-_{x+e_{k+1}, e_{k+1}} Z(A_k(x + e_{k+1}); \beta_k(x + e_{k+1}))]^{-\varepsilon_k(x)} \qquad (2.4.8)\end{aligned}$$

If $x \in V_{k+1}$, $Z(A_{k+1}(x); \beta_{k+1}(x))$ has to be replaced by $Z(A_{k+1}(x) \backslash \{x\};$ $\beta_{k+1}(x) \vee \alpha_{k+1}(x))$ in Eq. (2.4.8).

2.5. The Main Result

Now we can state one of the two main propositions of this section, namely the one which corresponds to the "polymerization."

Proposition 2.5.1. The following formula holds:

$$\begin{aligned}Z_{A_m} = & \prod_{j=1}^{2^d} \left\{ \prod_{x_j \in V_j} [Z(A_{2^d}(x_j), 0)]^{\varepsilon_{2^d}(x_j)} \right\} \\ & \times v^{2^d} \circ v^{2^d-1}_{\alpha > 2^d-1} \circ \cdots \circ v^2_{\alpha > 2} \left(\prod_{k=1}^{2^d} \left\{ \prod_{j \leqslant 2^{d(k)}} \left[\prod_{x_j \in V_j} (1 + \Phi^k_{x_j}) \right] \right\} \right) \qquad (2.5.1)\end{aligned}$$

The proof of Proposition 2.5.1 will be given in Section 3.

Now we are ready to write our partition function in terms of a gas of polymers whose only interaction is a hard-core exclusion.

For $k \in \{1,..., 2^d\}$ and $j \leqslant 2^{d(k)}$ let us define the following family of points:

$$C_0^k(x) = \begin{cases} B_k(x) & \text{if } \ x \in \Gamma_k \\ B_{k+1}(x) & \text{if } \ x \in \Gamma_k + e_{k+1} \end{cases} \qquad (2.5.2)$$

We define also

$$C_1^k(x) = \bigcup_{\varepsilon_1 \in \{-1, +1\}} \{ y \in \partial(x + \varepsilon_1 e_{k+1}),\ p(y) > k+1 \} \qquad (2.5.3)$$

and more generally, if l is an integer, $1 < l < 2^d - k$,

$$C_l^k(x) = \bigcup_{\varepsilon_1,..., \varepsilon_l \in \{-1, +1\}^l} \{ y \in \partial(x + \varepsilon_1 e_{k+1} + \cdots + \varepsilon_l e_{k+l}),\ p(y) > k + l \} \qquad (2.5.4)$$

For a given $k \in 1,..., 2^d$ and $x \in \mathbb{Z}^d$, we call a C^k bond the following set of points:

$$C^k(x) = \bigcup_{l=0}^{2^d-k} C_l^k(x)$$

A bond l will be always a $C^k(x)$ bond for some $k \in 1,..., 2^d$ and $x \in \mathbb{Z}^d$. We say that two bonds l_1 and l_2 are connected if $l_1 \cap l_2 \neq \emptyset$. A polymer R is a set of bonds $l_1,..., l_p$ that is connected in the following sense: $\forall ij$, $1 \leqslant i < j \leqslant p$, there exists a chain of connected bonds in R joining l_i to l_j. For $R = l_1,..., l_p$, we set $|R| = p$. The support $\tilde{R}$ of a polymer $R = l_1,..., l_p$ is $\tilde{R} = \bigcup_{i=1}^{p} l_i$.

We call $\mathscr{R}$ the set of all possible polymers with arbitrary support in $\mathbb{Z}^d$. $\mathscr{R}_\Lambda$ is the set of all polymers such that $\tilde{R} \subset \Lambda$. Two polymers R_i, R_j are said to be compatible if $\tilde{R}_i \cap \tilde{R}_j = \emptyset$; otherwise they are called incompatible.

Given a polymer $R \subset \mathscr{R}$: $R = C^{k_1}(x_1),..., C^{k_p}(x_p)$, we call the activity of R the quantity

$$\zeta(R) = v^{2^d} \circ v^{2^d-1}_{\alpha > 2^d - 1} \circ \cdots \circ v^2_{\alpha > 2} \left(\prod_{j=1}^{p} \Phi^{k_j}_{x_j} \right) \tag{2.5.5}$$

We can write

$$\frac{Z_{\Lambda_m}}{\prod_{j=1}^{2^d} \prod_{x_j \in V_j} [Z(\Lambda_{2^d}(x_j); 0)]^{\varepsilon_{2^d}(x_j)}} = 1 + \sum_{n \geqslant 1} \sum_{\substack{R_1,..., R_n \in \mathscr{R}_{\Lambda_m} \\ \tilde{R}_i \cap \tilde{R}_j = \emptyset}} \prod_{i=1}^{n} \zeta(R_i) \tag{2.5.6}$$

Then it is sufficient to divide our partition function by a suitable product of local terms in order to get the partition function of a gas of polymers whose only interaction is a hard-core exclusion. The denominator on the left-hand side of Eq. (2.5.6) represents the partition function of the "reference system," which is then given by a suitable set of noninteracting finite-volume systems.

Proposition 2.5.2. Let

$$\Xi_\Lambda = 1 + \sum_{n \geqslant 1} \sum_{\substack{R_1,..., R_n \in \mathscr{R}_\Lambda \\ \tilde{R}_i \cap \tilde{R}_j = \emptyset}} \prod_{i=1}^{n} \zeta(R_i) \tag{2.5.7}$$

Let

$$\varphi^T(R_1,..., R_n) = \frac{1}{n!} \sum_{g \in G(R_1,..., R_n)} (-1)^{(\# \text{edges in } g)} \tag{2.5.8}$$

where $G(R_1,..., R_n)$ is the set of connected graphs with n vertices $(1,..., n)$ and edges i, j corresponding to pairs R_i, R_j such that $\tilde{R}_i \cap \tilde{R}_j \neq \emptyset$ with the convention that if G is empty, the sum is equal to zero, and if $n = 1$, the sum is one. If

$$|\zeta(R)| \leqslant \lambda^{|R|} \tag{2.5.9}$$

with

$$\lambda < [3(2^{d+1}+1)]^{-d} 2^{-2d} e^{-4} \tag{2.5.10}$$

then (i) there exists a positive constant $C(\lambda, d)$ such that

$$\sum_{\substack{R_1,\dots,R_n \in \mathscr{R} \\ \exists R_i = R}} \varphi^T(R_1,\dots,R_n) \prod_{i=1}^{n} |\zeta(R_i)| \leqslant C(\lambda, d)\, e^{|R|}\, |\zeta(R)| \tag{2.5.11}$$

and (ii) in addition,

$$\Xi_\Lambda = \exp\left[\sum_{n\geqslant 1} \sum_{\substack{R_1,\dots,R_n \\ \tilde{R}_i \subset \Lambda}} \varphi^T(R_1,\dots,R_n) \prod_{i=1}^{n} \zeta(R_i)\right] \tag{2.5.12}$$

Proof. The proof can be obtained by the standard methods of the cluster expansion. We follow the lines of refs. 7 and 9, to which we refer for details. Let N be the maximal cardinality of a bond $C^k(x)$. For any $\alpha > 1$, we get, from Eq. (2.5.9),

$$\begin{aligned} |\zeta(R)| &\leqslant (\alpha\lambda)^{|R|} \left[\left(\frac{1}{\alpha}\right)^{1/N}\right]^{NR} \\ &\leqslant \left[\left(\frac{1}{\alpha}\right)^{1/N}\right]^{|\tilde{R}|} (\alpha\lambda)^{|R|} \end{aligned} \tag{2.5.13}$$

We can write, as in ref. 7,

$$|\zeta(R)| \leqslant \sigma^{|\tilde{R}|} \prod_{c \in R} \varphi_c \tag{2.5.14}$$

with

$$\sigma = \left(\frac{1}{\alpha}\right)^{1/N}, \qquad \varphi_c = \alpha\lambda \tag{2.5.15}$$

We have

$$\sum_{C \ni 0} \varphi_c \leqslant NM\alpha\lambda \equiv S \tag{2.5.16}$$

where M is the number of different bonds and we easily get $M \leqslant 2^d \cdot 2^d$.

Let us remark that if E is an upper bound for

$$\sup_{k=1,\dots,2^d} \sup_{x \in \mathbb{Z}^d} \operatorname{diam} C^k(x)$$

then

$$N < (3E)^d \tag{2.5.17}$$

On the other hand, it is easy to check that diam $C_0^1(0) \geqslant$ diam $C_0^k(x)$, $\forall x \in \mathbb{Z}^d$, $\forall k = 1,\dots, 2^d$, and, since diam $C_l^k \leqslant 2 +$ diam C_{l+1}^k for $l = 0,\dots, 2^d - 1$, we get diam $C^k(x) \leqslant 2^{d+1} + 1$. Thus, we can choose $E = 2^{d+1} + 1$. Now, by Lemma 1 of ref. 7, we get the result if

$$\exp S < \frac{1}{[\sqrt{\sigma}\,(2 - \sqrt{\sigma})]} \tag{2.5.18}$$

namely

$$\exp(NM\alpha\lambda) < \frac{\alpha^{1/2N}}{\alpha - (1/\alpha)^{1/2N}} \tag{2.5.19}$$

By choosing $\alpha = e^2$, one easily gets the final sufficient condition for the convergence:

$$\lambda[3(2^{d+1} + 1)]^d\, 2^{2d} < \frac{1}{e^4} \quad \blacksquare \tag{2.5.20}$$

Proposition 2.5.3. If $\exists L$ such that condition C_L of the introduction is satisfied, then, for the corresponding system of polymers, the following estimate holds:

$$|\zeta(R)| \leqslant \lambda^{|R|} \tag{2.5.21}$$

with

$$\lambda = [3(2^{d+1} + 1)]^{-d}\, 2^{-2d} e^{-4} \tag{2.5.22}$$

Proof. We write

$$\lambda_0 = \sup_{k = 1,\dots, 2^d} \; \sup_{j = 1,\dots, 2^{d(k)}} \; \sup_{\sigma(Q(C_0^k(x_j)))} |\Phi_{x_j}^k|$$

where x_j is any point in $\mathbb{Z}_j^d$ (by translation invariance, $\Phi_x^k = \Phi_y^k\ \forall x, y \in \mathbb{Z}_j^d$, $\forall j = 1,\dots, 2^d$).

Since all the $\nu_{\alpha > k}^k$ are normalized measures, we immediately get the result with λ_0 in place of λ.

For $x \in \mathbb{Z}^d$, let $\Lambda \subset D(x)$ and let $Z(\Lambda; \beta)$ be the partition function in $Q(\Lambda)$ with given boundary conditions $\beta \in S_{Q(D(x)) \setminus Q(\Lambda)}$. For a given unit vector e a straightforward calculation shows that

$$\frac{Z(\Lambda;\beta)\,S^0_{x,e}Z(\Lambda;\beta)}{S^+_{x,e}Z(\Lambda;\beta)\,S^-_{x,e}Z(\Lambda;\beta)}$$

$$= \sum_{\substack{\sigma_- \in S_{Q(\Lambda\cap Y(x-e;e))}\\ \sigma_+ \in S_{Q(\Lambda\cap Y(x+e;e))}}} \mu^{\beta_-\beta_0}_{\Lambda_-}(\sigma_-)\,\mu^{\beta_+\beta_0}_{\Lambda_+}(\sigma_+)$$

$$\times \frac{Z(\Lambda_0;\beta_-\vee\sigma_-,\beta_0,\beta_+\vee\sigma_+)\,Z(\Lambda_0;0,\beta_0,0)}{Z(\Lambda_0;\beta_-\vee\sigma_-,\beta_0,0)\,Z(\Lambda_0;0,\beta_0,\beta_+\vee\sigma_+)} \tag{2.5.23}$$

where

$$\beta_\pm = \beta\,|_{Y(x\pm e;e)}, \qquad \beta_0 = \beta\,|_{Y(x;e)}$$

$$\Lambda_\pm = \Lambda\cap Y(x\pm e;e), \qquad \Lambda_0 = \Lambda\cap Y(x;e)$$

and $\mu^{\beta_\pm\beta_0}_{\Lambda_\pm}(\sigma_\pm)$ are normalized Gibbs measures in $\Lambda_\pm$ with boundary conditions $\beta_-\beta_0$ and $\beta_+\beta_0$, respectively, from which, if C_L is true, one immediately gets the result.

Condition C'_L. For a given L, let

$$\lambda' = \sup_{k=1,\ldots,2^d}\ \sup_{j=1,\ldots,2^{d(k)}} \Bigg[\prod_{x_{2^d}\in C^k(x_j)\cap\mathbb{Z}^d_{2^d}}\ \sum_{\alpha_{2^d}(x_{2^d})} v^{2^d}(\alpha_{2^d}(x_{2^d}))\cdots$$

$$\times \prod_{x_{k+1}\in C^k(x_j)\cap\mathbb{Z}^d_{k+1}}\ \sum_{\alpha_{k+1}(x_{k+1})} v^{k+1}_{\alpha>k+1}(\alpha_{k+1}(x_{k+1}))\,|\Phi^k_{x_j}|^p\Bigg]^{1/p} \tag{2.5.24}$$

where x_j is any point of $\mathbb{Z}^d_j$, and p is an upper bound for the maximum number of C^k bonds that can pass through a point. A possible choice is

$$p = 2^{2d}[3(2^{d+1}+1)]^d \tag{2.5.25}$$

We say that condition C'_L is satisfied if

$$\lambda' \leqslant [3(2^{d+1}+1)]^{-d}\,2^{-2d}e^{-4} \tag{2.5.26}$$

Proposition 2.5.4. If $\exists L$ such that condition C'_L is satisfied, then for the corresponding system of polymers the following estimate holds:

$$|\zeta(R)| \leqslant \lambda^{|R|} \tag{2.5.27}$$

with

$$\lambda = [3(2^{d+1}+1)]^{-d}\,2^{-2d}e^{-4} \tag{2.5.28}$$

Proof. The proof will be given in the Appendix.

Remark 2.1. Following Dobrushin and Shlosman,[6] we define for $\Delta \subset V \subset \mathbb{Z}^d$, $|V| < \infty$, $\beta \in S_{\partial_{r_0} V}$,

$$q_{V,\Delta}(\sigma(\Delta) \mid \beta) = \sum_{\sigma(V \setminus \Delta)} \mu_V^\beta(\sigma(V)) \tag{2.5.29}$$

[see Eq. (1.3)].

Moreover, if q_1 and q_2 are probability measures on S_Δ, we set

$$\mathrm{Var}(q_1, q_2) = \frac{1}{2} \sum_{\sigma(\Delta)} |q_1(\sigma(\Delta)) - q_2(\sigma(\Delta))|$$

Let β_1, β_2 be configurations in $\partial_{r_0} V$ such that

$$\beta_1(x) = \beta_2(x), \qquad \forall x \neq t \in \partial_{r_0} V \tag{2.5.30}$$

Now suppose that the following condition $C_L^{D,S}$ is satisfied.

Condition $C_L^{D,S}$. For some positive k and γ we can find an L such that, if V is any subset of $P_{L,j}$, for some $j \in \{1, 2,..., d\}$ we have, $\forall \Delta \subset V$,

$$\mathrm{Var}(q_{V,\Delta}(\cdot \mid \beta_1), q_{V,\Delta}(\cdot \mid \beta_2)) \leqslant K \exp[-\gamma \operatorname{dist}(t, \Delta)] \tag{2.5.31}$$

Then, from Corollary 3.2, Eqs. (3.9) and (3.14) of ref. 1, it is easy to check that

$$\begin{aligned} \sup_{\sigma_-, \sigma_+, \tau, j} &\left| \frac{Z^{(j)}(\Lambda; \sigma_-, \sigma_+, \tau)\, Z^{(j)}(\Lambda; 0, 0, \tau)}{Z^{(j)}(\Lambda; \sigma_-, 0, \tau)\, Z^{(j)}(\Lambda; 0, \sigma_+, \tau)} - 1 \right| \\ &\leqslant 2e^{4\|u\|} (3L)^{2(d-1)}\, r_0^2 K e^{-\gamma L} \end{aligned} \tag{2.5.32}$$

where

$$\|u\| = \sup_{x \ni 0} \sup_{\sigma(x)} |u(\sigma(x))| \cdot \frac{1}{T}$$

In ref. 6 the authors introduce, among others, the following.

Constructive Condition III.c.[6] For some positive K and γ, for any region $V \subset \mathbb{Z}^d$ such that $(\operatorname{diam} V)/3 < L(K, \gamma, d)$, $\forall \Delta \subset V$, $\forall \beta_1, \beta_2$ satisfying (2.5.30), Eq. (2.5.31) is satisfied.

Of course, the above condition implies condition $C_L^{D,S}$.

Then, if, for a given potential U, Condition IIIc of ref. 6 is satisfied with $L(K, \gamma, d)$ given by

$$2e^{4\|u\|} r_0^2 (3L)^{2(d-1)} K e^{-\gamma L} [3(2^{d+1} + 1)]^d\, 2^{2d} e^4 < 1 \tag{2.5.33}$$

our Theorem 1.1 implies that U is completely analytic (see ref. 6).

We have thus obtained an alternative proof of Theorem 4.2 of ref. 6. We recall that the condition ref. 6 [Eq. (4.1)] analogous to our (2.5.32) is

$$[(K+1)(3L+2r_0+1)^d]^{2d+3}\, 2d \exp\left(-\frac{\gamma L}{3d}\right) < 1 \tag{2.5.34}$$

3. PROOF OF PROPOSITION 2.5.1

We will prove Proposition 2.5.1 by recurrence on the number of steps that we have performed.

Let us define a quantity $Z_\Lambda(t)$, where t is an integer variable which can be considered as a time, by the following formula if $t \in 2,..., 2^d$:

$$\begin{aligned} Z_\Lambda(t=k) = {} & \sum_{\alpha_{2^d}} e^{H_V(\alpha_{2^d})} \sum_{\alpha_{2^d-1}} e^{H_V(\alpha_{2^d-1}) + W_V(\alpha_{2^d-1}, \alpha_{2^d})} \cdots \\ & \times \sum_{\alpha_{k+1}} e^{H_V(\alpha_{k+1}) + W_V(\alpha_{k+1}, \alpha_{>k+1})} \\ & \times \prod_{x \in V} [Z(A_k(x); \beta_k(x)]^{\varepsilon_k(x)} \\ & \times v^k_{\alpha_{>k}} \circ v^{k-1}_{\alpha_{>k-1}} \circ \cdots \circ v^2_{\alpha_{>2}} \left[\prod_{k'=1}^{k-1} \prod_{j \leqslant 2^{d(k')}} \prod_{x_j \in V_j} (1+\Phi^{k'}_{x_j}) \right] \end{aligned} \tag{3.0}$$

where the set $A_k(x)$ is defined in Eqs. (2.4.1) and (2.4.4) and the boundary condition $\beta_k(x)$ is defined according to Eqs. (2.4.2) and (2.4.5), where the generic spin configuration σ is nothing but $\alpha_{k+1},..., \alpha_{2^d}$.

In particular, it follows immediately from Eq. (2.3.11) that $Z_\Lambda(t=2) = Z_\Lambda$. Now it is clear that, if we prove that $Z_\Lambda(t=k) = Z_\Lambda$ implies $Z_\Lambda(t=k+1) = Z_\Lambda$, we have proved by recurrence Proposition 2.5.1 by taking $k = 2^d$.

We start by proving some lemmas.

Lemma 3.1. For any $x \in \Gamma_k$ we have

$$S^0_{x, e_{k+1}} Z(A_k(x); \beta_k(x)) = Z(A_{k+1}(x); \beta_{k+1}(x)) \tag{3.1}$$

Proof. We have first to prove that $S^0_{x, e_{k+1}} A_k(x) = A_{k+1}(x)$, namely

$$A_k(x) \cap Y(x, e_{k+1}) = A_{k+1}(x)$$

Let x_k be a point of $\mathbb{Z}^d_k$ such that $x \in \partial x_k$. From Definition 2.4.1, since $x_k + e_{k+1} \equiv x_{k+1} \in \mathbb{Z}^d_{k+1}$ and $Y(x; e_{k+1}) \not\ni x_{k+1}$, we immediately get

$$A_k(x_k) \cap Y(x; e_{k+1}) = A_{k+1}(x_{k+1}) \cap Y(x; e_{k+1}) \tag{3.2}$$

Let $j=p(x)$; it is clear that we can find $\gamma(k, j)$ and $\gamma(k+1, j)$ such that (i) $\gamma(k+1, j)= -e_{k+1}+\gamma(k, j)$ and (ii) $\gamma(k, j) \perp e_{k+1}$.

By intersecting both sides of Eq. (3.2) with $Y(x, \gamma(k, j))$ and using Definition 2.4.3, we get

$$A_k(x) \cap Y(x; e_{k+1}) = A_{k+1}(x)$$

To prove $S^0_{x,e_{k+1}}\beta_k(x)=\beta_{k+1}(x)$, we proceed in a similar way starting from Eqs. (2.4.2) and (2.4.5).

Lemma 3.2. For any $x \in \Gamma_k$ we have

$$S^+_{x,e_{k+1}}A_k(x) \cap S^-_{x+2e_{k+1},e_{k+1}}A_k(x+2e_{k+1}) = A_k(x+e_{k+1}) \tag{3.3}$$

In particular, if $p(x+e_{k+1}) > 2^{d(k)}$,

$$S^+_{x,e_{k+1}}A_k(x) \cap S^-_{x+2e_{k+1},e_{k+1}}A_k(x+2e_{k+1}) = \varnothing \tag{3.4}$$

Moreover, we have

$$\varepsilon_k(x+e_{k+1}) = -\varepsilon_k(x) = -\varepsilon_k(x+2e_{k+1}) \tag{3.5}$$

Finally,

$$S^+_{x,e_{k+1}}\beta_k(x) \qquad \text{and} \qquad S^-_{x+2e_{k+1},e_{k+1}}\beta_k(x+2e_{k+1}) \tag{3.6}$$

coincide on $Y(x+e_{k+1}; e_{k+1})$.

Proof. Let us prove (3.3). Since $x \in \Gamma_k$, it is easy to check that

$$Y(x+e_{k+1}; e_{k+1}) = Y(x_k+e_{k+1}; e_{k+1}) \tag{3.7}$$

where $x_k \in \mathbb{Z}^d_k \subset \Gamma_k$ is such that $x \in \partial x_k$. Moreover, if e is any unit vector orthogonal to e_{k+1}, we have

$$Y(x, e) = Y(x+e_{k+1}; e) = Y(x+2e_{k+1}; e) \tag{3.8}$$

Therefore, if $j=p(x)$, since any $\gamma(k, j)$ is orthogonal to e_{k+1}, we get

$$Y(x; \gamma(k, j)) = Y(x+e_{k+1}; \gamma(k, j)) = Y(x+2e_{k+1}; \gamma(k, j)) \tag{3.9}$$

Consider the case $x=x_k \in \mathbb{Z}^d_k$, since

$$D(x_k+2e_{k+1}) \cap Y(x_k+e_{k+1}; e_{k+2}) = D(x_k) \cap Y(x_k+e_{k+1}; e_{k+2})$$

from Eq. (2.4.1) one gets

$$A_k(x_k) \cap Y(x_k+e_{k+1}; e_{k+2}) = A_k(x_k+2e_k) \cap Y(x_k+e_{k+2}; e_{k+1})$$

And then, since

$$\begin{aligned} &S^{+}_{x_k, e_{k+2}} A_k(x_k) \cap S^{-}_{x_k+2e_{k+1}, e_{k+2}} A_k(x+2e_{k+1}) \\ &\quad \equiv A_k(x_k) \cap Y(x_k+e_{k+1}; e_{k+1}) \cap A_k(x_k+2e_{k+1}) \\ &\quad = A_k(x_k) \cap Y(x_k+e_{k+1}; e_{k+1}) \end{aligned} \tag{3.10}$$

Eq. (3.3) is proven for $x \in \mathbb{Z}^d_k$. Now, from Eqs. (3.7), (3.9), and (2.4.4) we get the proof of Eq. (3.3) for the general case.

Notice that if $p(x_k+e_{k+1}) > 2^{d(k)}$, then both side of Eq. (3.10) are empty.

By similar arguments we prove (3.6). Finally, Eq. (3.5) is a direct consequence of the definition of $\varepsilon_k(x)$.

Lemma 3.3. For any $x \in \Gamma_k$, $p(x) \neq k$, we have

$$S^{+}_{x, e_{k+1}} A_k(x) \cup S^{-}_{x+2e_{k+1}, e_{k+1}} A_k(x+2e_{k+1}) = A_{k+1}(x+e_{k+1}) \tag{3.11}$$

$$S^{+}_{x, e_{k+1}} \beta_k(x) \vee S^{-}_{x+2e_{k+1}, e_{k+1}} \beta_k(x+2e_{k+1}) = \beta_{k+1}(x+e_{k+1}) \tag{3.12}$$

If $p(x)=k$, i.e., $x = x_k \in \mathbb{Z}^d_k$,

$$\begin{aligned} &S^{+}_{x, e_{k+1}} A_k(x) \cup S^{-}_{x+2e_{k+1}, e_{k+1}} A_k(x+2e_{k+1}) \\ &\quad = A_{k+1}(x+e_{k+1}) \backslash \{x+e_{k+1}\} \end{aligned} \tag{3.13}$$

$$\begin{aligned} &S^{+}_{x, e_{k+1}} \beta_k(x) \vee S^{-}_{x+2e_{k+1}, e_{k+1}} \beta_k(x+2e_{k+1}) |_{\partial(x+e_{k+1})} \\ &\quad = \beta_{k+1}(x+e_{k+1}) \end{aligned} \tag{3.14}$$

Proof. We have already proven [see Eq. (3.2)] that for $x \in \Gamma_k$

$$A_{k+1}(x_{k+1}) \cap Y(x; e_{k+1}) = A_k(x_k) \cap Y(x; e_{k+1}) \tag{3.15}$$

and by a similar argument we have also

$$\begin{aligned} &A_{k+1}(x_k+e_{k+1}) \cap Y(x+2e_{k+1}; e_{k+1}) \\ &\quad = A_k(x_k+2e_{k+1}) \cap Y(x+2e_{k+1}; e_{k+1}) \end{aligned} \tag{3.16}$$

It follows from Eq. (2.4.1) that

$$\begin{aligned} &[A_k(x_k) \cup A_k(x_k+2e_{k+1})] \cap Y(x_k+e_{k+1}; e_{k+1}) \\ &\quad = [A_{k+1}(x_k+e_{k+1}) \cap Y(x_k+e_{k+1}; e_{k+1})] \backslash \{x_k+e_{k+1}\} \end{aligned} \tag{3.17}$$

Now putting together Eqs. (3.15)–(3.17), one gets Eq. (3.13). Equation (3.11) follows from Eq. (3.13) and (3.9). By similar arguments, one gets the statements (3.12) and (3.14) relative to the boundary conditions.

Let us first consider the case where $p(x_k + e_{k+1}) \leqslant 2^{d(k)}$. We start by writing, for $x \in \Gamma_k$, the following identiy corresponding to the unfolding:

$$\begin{aligned}
&[Z(A_k(x); \beta_k(x))]^{\varepsilon_k(x)} \\
&\quad = (1 + \Phi_x^k) \\
&\quad\quad \times \left[\frac{[S^+_{x, e_{k+1}} Z(A_k(x); \beta_k(x))][S^-_{x, e_{k+1}} Z(A_k(x); \beta_k(x))]}{S^0_{x, e_{k+1}} Z(A_k(x); \beta_k(x))} \right]^{\varepsilon_k(x)}
\end{aligned} \tag{3.18}$$

where Φ_x^k is defined in Eq. (2.4.7).

Then we perform a splitting, namely we associate the term

$$(1 + \Phi_x^k)[S^0_{x, e_{k+1}} Z(A_k(x); \beta_k(x))]^{\varepsilon_k(x)} \tag{3.19}$$

to $x \in \Gamma_k$ and, for $\varepsilon = \{-1, 1\}$, we associate the term

$$[S^\varepsilon_{x, e_{k-1}} Z(A_k(x); \beta_k(x))]^{\varepsilon_k(x)} \tag{3.20}$$

to $x + \varepsilon e_{k+1}$. Notice that $x + \varepsilon e_{k+1} \in \Gamma_k$. Therefore, after splitting, the following quantities are associated to $x \in \Gamma_k + e_{k+1}$:

$$[Z(A_k(x); \beta_k(x))]^{\varepsilon_k(x)} \tag{3.21}$$

already present, and

$$\begin{aligned}
&[S^+_{x - e_{k+1}, e_{k+1}} Z(A_k(x - e_{k+1}); \beta_k(x - e_{k+1}))]^{\varepsilon_k(x - e_{k-1})} \\
&[S^-_{x + e_{k+1}, e_{k+1}} Z(A_k(x + e_{k+1}); \beta_k(x + e_{k+1}))]^{\varepsilon_k(x + e_{k+1})}
\end{aligned} \tag{3.22}$$

coming from the splitting in the direction e_{k+1} at $x \pm e_{k+1}$.

Now we deduce from Lemma 3.2 that the expression that we have so far is such that we can perform the gluing on $\Gamma_k + e_{k+1}$.

Let us write the following identities, which, by Lemma 3.3 and Eq. (3.5), since $\varepsilon_k(x) = -\varepsilon_{k+1}$, correspond to gluing. For $x \in \Gamma_k + e_k$: $p(x) \neq k+1$,

$$\begin{aligned}
&[Z(A_k(x); \beta_k(x))]^{\varepsilon_k(x)} \\
&\qquad \times [S^+_{x - e_{k+1}, e_{k+1}} Z(A_k(x - e_{k+1}); \beta_k(x - e_{k+1})) \\
&\qquad \times S^-_{x + e_{k+1}, e_{k+1}} Z(A_k(x + e_{k+1}); \beta_k(x + e_{k+1}))]^{\varepsilon_k(x)} \\
&\quad = [Z(A_{k+1}(x), \beta_{k+1}(x))]^{\varepsilon_{k+1}(x)} (1 + \Phi_x^k)
\end{aligned} \tag{3.23}$$

For x: $p(x)=k+1$

$$[Z(A_k(x), \beta_k(x))]^{\varepsilon_k(x)}$$
$$\times [S^+_{x-e_{k+1},e_{k+1}} Z(A_k(x-e_{k+1}); \beta_k(x-e_{k+1}))$$
$$\times S^-_{x+e_{k+1},e_{k+1}} Z(A_k(x+e_{k+1}); \beta_k(x+e_{k+1}))]^{\varepsilon_k(x)}$$
$$= [Z(A_{k+1}(x)\backslash\{x\}; \beta_{k+1}(x) \vee \alpha_{k+1}(x))]^{\varepsilon_{k+1}(x)} (1+\Phi^k_x) \qquad (3.24)$$

Now it is immediate to check that in all the sublattices $\mathbb{Z}^d_j$, $j \leqslant 2^{d(k)}$, $j \neq k+1$, the new partition functions we have obtained after unfolding on Γ_k and splitting and gluing on $\Gamma_k + e_{k+1}$ are the ones corresponding to $Z_A(t=k+1)$.

We notice at this point that these last partition functions do not depend on the configuration variables α_{k+1}. This follows from the following facts:

1. After the splitting at $x \in \Gamma_k$ there are boundary conditions $S^0_{x,e_{k+1}}(\beta_k(x))$ which (see Definition 2.2.1) do not contain α_{k+1} in the sense that

$$\beta_k(x)|_{\Gamma_k+e_{k+1}} = 0 \qquad \text{and then} \quad \beta_k(x)|_{V_{k+1}} = 0$$

2. At any $x \in \Gamma_k + e_{k+1}$ with $p(x) \neq k+1$, by Lemma 3.3 there are boundary conditions $\beta_{k+1}(x)$.

Looking at Definition 2.4.3, we see that $\beta_{k+1}(x)|_{V_{k+1}} = 0$ and so also $\beta_{k+1}(x)$ do not contain α_{k+1}. We conclude that at this point the only dependence on $\alpha_{k+1}(x)$, for $x \in V_{k+1}$, besides the one present in the error terms, is given by the unnormalized (Bernoulli) measure:

$$\tilde{v}_{\alpha_{>k+1}}(\alpha_{k+1}(x))$$
$$= e^{H(\alpha_{k+1}(x)) + W(\alpha_{k+1}(x), \alpha_{>k+1})} Z(A_{k+1}(x)\backslash\{x\}; \beta_{k+1}(x) \vee \alpha_{k+1}(x) \qquad (3.25)$$

We can write

$$Z_A(t=k) = \sum_{\alpha_{2^d}} e^{H_A(\alpha_{2^d})} \sum_{\alpha_{2^d-1}} e^{H_A(\alpha_{2^d-1}) + W_A(\alpha_{2^d}, \alpha_{2^d-1})}$$
$$\times \cdots \sum_{\alpha_{k+2}} e^{H_A(\alpha_{k+2}) + W_A(\alpha_{k+2}, \alpha_{>k+2})}$$
$$\times \prod_{\substack{j \neq k+1 \\ j \leqslant 2^{d(k)}}} \prod_{x_j \in V_j} [Z(A_k(x_j); \beta_k(x_j))]^{\varepsilon_k(x_j)}$$
$$\times \sum_{\alpha_{k+1}} \prod_{x_{k+1} \in V_{k+1}} \tilde{v}_{\alpha_{>k+1}}(\alpha_{k+1}(x_{k+1}))$$
$$\times \prod_{j \leqslant 2^{d(k)}} \prod_{x_j \in V_j} [1 + \Phi^{k+1}_{x_j}(\alpha_{\geqslant k+1})]$$
$$\times v^k_{\alpha_{>k}} \circ \cdots \circ v^2_{\alpha_{>2}} \left(\prod_{k'=1}^{k-1} \prod_{j \leqslant 2^{d(k)}} \prod_{x_j \in V_j} (1 + \Phi^{k'}_{x_j}) \right) \qquad (3.26)$$

If we divide the expression (3.30) by the normalization factor

$$Z(A_{k+1}(x); \beta_{k+1}(x))$$

we get nothing but the measure $\nu_{\beta_{k+1}}^{k+1}(\alpha_{k+1}(x))$ (see Definition 2.21).

Then, to conclude the proof of Proposition 2.5.1 for the case $d(k+1)=d(k)$ it is sufficient to multiply and divide the rhs of Eq. (3.31) by

$$\prod_{x_{k+1}\in V_{k+1}} Z(A_{k+1}(x_{k+1}); \beta_{k+1}(x_{k+1})) \tag{3.27}$$

and finally to perform the sum over the α_{k+1} variables. Looking at expression (3.26), it is easy to convince oneself that $Z_A(t=k)=Z_A(t=k+1)$.

The case when $p(x_k+e_{k+1})>2^{d(k)}$ involves only minor changes and is left to the reader.

APPENDIX. PROOF OF PROPOSITION 2.5.4

Let $R=C^{k_1}(x_1),..., C^{k_q}(x_q)$, $q=|R|$, be a generic polymer; then if $p\equiv p(d)$ is given by Eq. (2.5.25), we want to prove the following inequality:

$$|\zeta(R)|\leqslant \prod_{j=1}^{q} [\nu^{2^d}\circ \cdots \circ \nu_{\alpha>k_j}^{k_j}(|\Phi_{x_j}^{k_j}|^p)]^{1/p}$$

from which we get, using C'_L the desired result:

$$|\zeta(R)|\leqslant \{[3(2^{d+1}+1)]^{-d}2^{-2d}e^{-4}\}^{|R|}$$

We write $I=\{1,..., q\}$ and, if $x\in\tilde{R}$,

$$I(x)=\{j\in I/\tilde{C}^{k_j}(x_j)\ni x\}$$
$$I(x^c)=I\backslash I(x)$$

Let us write

$$\xi(R)=\prod_{j\in I}|\Phi_{x_j}^{k_j}|$$
$$\xi(R,x)=\prod_{j\in I(x)}|\Phi_{x_j}^{k_j}|$$
$$\xi(R,x^c)=\prod_{j\in I(x^c)}|\Phi_{x_j}^{k_j}|$$

If $\tau=\tau(R)=\mathrm{Inf}(j\geqslant 2/\tilde{R}\cap\mathbb{Z}_j^d\neq 0)$, then we get

$$\nu^{2^d}\circ\cdots\circ\nu_{\alpha>2}^{2}(\xi(R))=\nu^{2^d}\circ\cdots\circ\nu_{\alpha>\tau}^{\tau}(\xi(R))$$

Let $\mathbb{Z}_\tau^d \cap \tilde{R} = \{y_1, ..., y_N\}$ with $N = |\mathbb{Z}_\tau^d \cap \tilde{R}|$. Since $v_{\alpha_{>\tau}}^\tau$ is a product measure, we can write

$$v_{\alpha_{>\tau}}^\tau(\xi(R)) = \prod_{\substack{x \in V_\tau \\ x \neq y_1}} \sum_{\alpha_\tau(x)} v_{\alpha_{>\tau}}^\tau(\alpha_\tau(x))(\xi(R, y_1^c)) \times \sum_{\alpha_\tau(y_1)} v_{\alpha_{>\tau}}^\tau(\alpha_\tau(y_1))\, \xi(R, y_1)$$

If we write

$$\Phi_{x_j, y_1}^{k_j} = \left[\sum_{\alpha_\tau(y_1)} v_{\alpha_{>\tau}}^\tau(\alpha_\tau(y_1))\, |\Phi_{x_j}^{k_j}|^p \right]^{1/p}$$

using the fact that $\xi(R, y_1)$ is a product of at most p terms $\Phi_{x_j}^{k_j}$ and the Hölder inequality, it is not difficult to check that

$$\sum_{\alpha_\tau(y_1)} v_{\alpha_{>\tau}}^\tau(\alpha_\tau(y_1))\, \xi(R, y_1) \leqslant \prod_{x \in V_\tau} \Phi_{x_j, y_1}^{k_j}$$

Writing

$$I(y_1, y_2) = I(y_1) \cap I(y_2)$$
$$I(y_1^c, y_2) = I(y_1^c) \cap I(y_2)$$
$$I(y_1^c, y_2^c) = I(y_1^c) \cap I(y_2^c)$$

and

$$\xi(R, y_1, y_2) = \prod_{j \in I(y_1, y_2)} |\Phi_{x_j, y_1)}^{k_j}|$$
$$\xi(R, y_1^c, y_2) = \prod_{j \in I(y_1^c, y_2)} |\Phi_{x_j}^{k_j}|$$
$$\xi(R, y_1^c, y_2^c) = \prod_{j \in I(y_1^c, y_2^c)} |\Phi_{x_j}^{k_j}|$$

we get

$$v_{\alpha_{>\tau}}^\tau(\xi(R)) \leqslant \prod_{x \in V_\tau \setminus \{y_1, y_2\}} \sum_{\alpha_\tau(x)} v_{\alpha_{>\tau}}^\tau(\alpha_\tau(x))\, \xi(R, y_1^c, y_2^c) \times \sum_{\alpha_\tau(y_2)} v_{\alpha_{>\tau}}^\tau(\alpha_\tau(y_2))\, \xi(R, y_1, y_2)\, \xi(R, y_1^c, y_2)$$

Since $|I(y_2)| \leqslant p$, we have that $\xi(R, y_1, y_2)\, \xi(R, y_1^c, y_2)$ is a product of at most p terms $\Phi_{x_j}^{k_j}$ or $\Phi_{x_j, y_1}^{k_j}$.

If we write

$$\Phi^{k_j}_{x_k, y_1, y_2} = \left[\sum_{\alpha_\tau(y_2)} v^\tau_{\alpha_{>\tau}}(\alpha_\tau(y_2)) \, |\Phi^{k_j}_{x_j}|^p \right]^{1/p}$$

if $j \in I(y_1^c, y_2)$,

$$\Phi^{k_j}_{x_j, y_1, y_2} = \left[\sum_{\alpha_\tau(y_2), \alpha_\tau(y_1)} v^\tau_{\alpha_{>\tau}}(\alpha_\tau(y_2)) \, v^\tau_{\alpha_{>\tau}}(\alpha_\tau(y_1)) \, |\Phi^{k_j}_{x_j}|^p \right]^{1/p}$$

if $j \in I(y_1, y_2)$, using the Hölder inequality, it is not difficult to check that

$$\sum_{\alpha_\tau(y_2)} v^\tau_{\alpha_{>\tau}}(\alpha_\tau(y_2)) \, \xi(R, y_1, y_2) \, \xi(R, y_1^c, y_2)$$
$$\leqslant \prod_{j \in I(y_2)} \Phi^{k_j}_{x_j, y_1, y_2}$$

By recurrence on the number of points in $\tilde{R} \cap \mathbb{Z}^d_\tau$, writing

$$\Phi^{k_j}_{x_j, \tau} = [v^\tau_{\alpha_{>\tau}}(|\phi^{k_j}_{x_j}|^p)]^{1/p}$$

we get

$$v^\tau_{\alpha_{>\tau}}(\xi(R)) \leqslant \prod_{j \in I} \phi^{k_j}_{x_j, \tau}$$

Now it is clear that we can iterate this procedure by summing over the configurations in $S_{V_{\tau+1}}, S_{V_{\tau+2}}, \ldots, S_{V_{2^d}}$, use again the Hölder inequality, and get the result.

ACKNOWLEDGMENTS

E. O. wishes to express his thanks to the Centre de Physique Théorique, CNRS, Luminy, Marseille for their warm hospitality.

P. P. thanks for its kind hospitality the Dipartimento di Matematica dell'Università di Roma "La Sapienza," where part of this work was done.

REFERENCES

1. E. Olivieri, On a cluster expansion from lattice spin systems: A finite size condition for the convergence, *J. Stat. Phys.* **50**:1179 (1988).
2. G. Gallavotti and S. Miracle-Solè, Correlation function of a lattice system, *Commun. Math. Phys.* **7**:274 (1968).
3. R. L. Dobrushin and S. B. Shlosman, Constructive criterion for the uniqueness of Gibbs field, in *Statistical Physics and Dynamical Systems*, D. Sasz and D. Petz, eds. (Birkäuser, Boston, 1985).

4. R. L. Dobrushin and S. B. Shlosman, Completely analytical Gibbs fields, in *Statistical Physics and Dynamical Systems*, D. Sasz and D. Petz, eds. (Birkäuser, Boston, 1985).
5. R. L. Dobrushin, Induction on volume and no cluster expansion, in *Proceedings of the VIIth IAMP, Marseille '86*, M. Mebkhout and R. Sénéor, eds. (World Scientific, Singapore, 1987).
6. R. L. Dobrushin and S. B. Shlosman, Completely analytical interactions: A constructive description, *J. Stat. Phys.* **46**:983 (1987).
7. M. Cassandro and E. Olivieri, Renormalization group and analyticity in one dimension: a Proof of Dobrushin's theorem, *Commun. Math. Phys.* **80**:255 (1981).
8. M. Campanino, M. Capocaccia, and E. Olivieri, Analyticity for one dimensional systems with long range superstable interactions, *J. Stat. Phys.* **33**:437 (1983).
9. G. Gallavotti, A. Martin Löf, and S. Miracle-Sole, Some problems connected with the description of coexisting phases at low temperatures in the Ising model, in *Lecture Notes in Physics*, Vol. 20, pp. 162–204.